

COUNTABLE EXCHANGE AND FULL EXCHANGE RINGS

PACE P. NIELSEN

ABSTRACT. We show that a suitable ring with a “nice” topology, in which convergent limits of units are units, is an \aleph_0 -exchange ring. We generalize the argument to show that a semi-regular ring, R , with a “nice” topology, is a full exchange ring. Putting these results in the language of modules, we show that a cohopfian module with finite exchange has countable exchange. Also, all modules with Dedekind-finite, semi-regular endomorphism rings are full exchange modules.

§1. INTRODUCTION

The exchange property for modules was first studied in 1964 by Crawley and Jónsson [CJ], and is defined as follows. A right k -module M_k has the \aleph -*exchange property* if, whenever $A = M \oplus N = \bigoplus_{i \in I} A_i$, with $|I| \leq \aleph$, then there are submodules $A'_i \subseteq A_i$, with $A = M \oplus (\bigoplus_{i \in I} A'_i)$. If M has \aleph -exchange for all cardinals \aleph then we say M has *full exchange*. If the same holds just for the finite cardinals, we say M has *finite exchange*. It is easy to show that 2-exchange is equivalent to finite exchange. An outstanding question in module theory is whether or not finite exchange further implies full exchange.

It turns out that the finite exchange property is an endomorphism ring invariant; putting $E = \text{End}(M_k)$, then M_k has finite exchange if and only if E_E has finite exchange. A ring, R , such that R_R has finite exchange is called an *exchange ring*, following [Wa], and this turns out to be a left-right symmetric condition. Nicholson [N] calls a ring *suitable* if, given an equation $x + y = 1$, there are orthogonal idempotents $e \in Rx$ and $f \in Ry$ with $e + f = 1$. This turns out to be equivalent to R being an exchange ring. It is easy to show that semi- π -regular rings¹ are suitable, and while this is a large class it does not exhaust all exchange rings. Any corner ring in a suitable ring is suitable, and any direct product of suitable rings is suitable.

Continuous modules, and hence (quasi-)injective modules, always claim the exchange property [MM₂]. Further, quasi-continuous modules with finite exchange have full exchange [OR], [MM₃]. There are many other classes of modules for which finite exchange implies full exchange, including modules which are direct sums of indecomposables [ZZ], and modules with abelian endomorphism rings [Ni]. It also turns out that square-free modules² with finite exchange have countable exchange [MM₁].

Every endomorphism ring, E , is endowed with a topology, called the *finite topology*, in which a basis of neighborhoods of zero is given by annihilators of finite subsets of

¹ $R/J(R)$ is π -regular, and idempotents lift modulo $J(R)$.

²No submodule is isomorphic to a square $X \oplus X$.

M . One says that a collection $\{x_i\}_{i \in I} \subseteq E$ of endomorphisms is *summable*, if for each $m \in M$ the set $\{i | x_i(m) \neq 0\}$ is finite. One may then easily define $\sum_{i \in I} x_i$ as the map $m \mapsto \sum_{i \in I} x_i(m)$. Central to the study of exchange modules is the following proposition:

Proposition 1. *The following are equivalent:*

- (1) M has the \aleph -exchange property.
- (2) If we have

$$A = M \oplus N = \bigoplus_{i \in I} A_i$$

with $A_i \cong M$ for all $i \in I$, and $|I| \leq \aleph$, then there are submodules $A'_i \subseteq A_i$ such that

$$A = M \oplus \bigoplus_{i \in I} A'_i.$$

- (3) Given a summable family $\{x_i\}_{i \in I}$ of elements of E , with $\sum_{i \in I} x_i = 1$, and with $|I| \leq \aleph$, then there are orthogonal idempotents $e_i \in Ex_i$ with $\sum_{i \in I} e_i = 1$.

Proof. This is [ZZ, Proposition 3]. □

Now, let R be a topological ring with a linear, Hausdorff topology. This means that there is a ring topology with a basis of zero, say \mathfrak{U} , consisting of left ideals, with $\bigcap_{U \in \mathfrak{U}} U = (0)$. We say that a collection $\{x_i\}_{i \in I} \subseteq R$ is *summable to* $r \in R$ if there is a finite set $F' \subseteq I$ such that $\sum_{i \in F} a_i - r \in U$ for all finite sets $F \supseteq F'$. The finite topology on E is linear and Hausdorff, and this new notion of summability agrees with the one defined above. Following [MM₁], we can now extract from Proposition 1 property (3) a ring theoretic version of \aleph -exchange.

Definition 1. Let R be a ring with a linear, Hausdorff topology. We say that R is an \aleph -exchange ring if, given a summable family $\{x_i\}_{i \in I} \subseteq R$ with $\sum_{i \in I} x_i = 1$, then there are summable, orthogonal idempotents $\{e_i\}_{i \in I}$ with $e_i \in Rx_i$ and $\sum_{i \in I} e_i = 1$.³ If this holds for all cardinalities \aleph , we say the ring is a *full exchange ring*.

Notice, a module has \aleph -exchange if and only if E (with the finite topology) is an \aleph -exchange ring. Also notice, in the definition above we require $\{e_i\}_{i \in I}$ to be a summable family. When trying to verify that a ring is an \aleph -exchange ring, we often need to assume some condition which forces families of this sort to be summable. The following is such a condition: We say a summable family $\{x_i\}$ is *left multiple summable* if, given an arbitrary family $\{r_i\}_{i \in I}$, then the collection $\{r_i x_i\}_{i \in I}$ is also summable. We say that a topology is *left multiple summable* if all summable families are left multiple summable. Finally, we say that a topological ring, R , has a *nice topology* if the topology is linear, Hausdorff, and left multiple summable. One can easily show that a complete, linear, Hausdorff topology is nice.

³This definition differs from the one given in [MM₁], where the word “orthogonal” is missing, and the word “complete” is added. From a personal correspondence with the first author of that paper, it was made clear that the definition given here is the one they intended.

In this paper, we show that a suitable ring with a nice topology, in which convergent limits of units are units, is an \aleph_0 -exchange ring. Generalizing the proof, we then show that Dedekind-finite, regular rings with nice topologies are full exchange rings. We generalize the proof further to show that π -regular, nice topological rings are full exchange rings, if the right regular module R_R satisfies the (C_2) property. Further, we push these arguments through the radical. We finish by reinterpreting these results in module-theoretic language.

§2. TOOLS FOR EXCHANGE RINGS

Throughout this paper we let k be a ring, we let M_k be a right k -module, and put $E = \text{End}(M_k)$, which acts on the left of M . All other modules will also be right k -modules. If we have two modules N and N' we write $N \subseteq^\oplus N'$ to mean that N is a direct summand of N' . Also throughout, we let R be a ring, $U(R)$ the group of units, and $J(R)$ the Jacobson radical. Rings are associative with 1, and modules are unital.

In our study of \aleph -exchange rings, we first investigate the behavior of idempotents in suitable rings. To begin, we define a useful equivalence relation on idempotents.

Definition 2. Let $e, e' \in R$ be idempotents. We say that e and e' are *left strongly isomorphic* if $e'e = e'$ and $ee' = e$. We write this relation as $e \sim e'$, and it is easy to check that this is an equivalence relation. One also has the dual notion of right strongly isomorphic idempotents, which we denote by $e \smile e'$.

Lemma 1. *Let e and e' be idempotents in a ring R . The following are equivalent:*

- (1) $e \sim e'$.
- (2) $Re = Re'$.
- (3) $e' = e + (1 - e)re$ for some $r \in R$.
- (4) $e' = ue$ for some $u \in U(R)$.
- (5) $(1 - e) \smile (1 - e')$.

Furthermore, if $R = \text{End}(M_k)$ for some module M_k , then the following properties are also equivalent to the ones above:

- (6) $\ker(e) = \ker(e')$.
- (7) $(1 - e)M = (1 - e')M$.

Proof. The equivalence of properties (1) through (5) is a simple exercise [La₂, Exercise 21.4]. (6) \Leftrightarrow (7) is easy, as is (1) \Leftrightarrow (6). \square

In the literature, two idempotents e, e' are said to be *isomorphic* if $eR \cong e'R$ (or equivalently, $Re \cong Re'$). Thus, we see that if two idempotents are left (or right) strongly isomorphic then they are isomorphic. On the other hand, two idempotents are both left and right strongly isomorphic if and only if they are equal. So, the notion of left strongly isomorphic idempotents is a nontrivial strengthening of the notion of isomorphic idempotents.

The equivalence in Lemma 1 that we need the most is $(1) \Leftrightarrow (4)$. It turns out that we can say more about the unit in property (4). In fact, by property (3), $e' = e + (1 - e)re$ for some $r \in R$. Putting $u = 1 + (1 - e)re$, we see that $e' = ue$, and u is a unit with inverse $u^{-1} = 1 - (1 - e)re$. Also notice, $u(1 - e) = (1 - e)$. So, we may strengthen property (4) to read:

$$(4') \quad e' = ue \text{ for some } u \in U(R), \text{ with } u(1 - e) = (1 - e).$$

Throughout the rest of the paper, we will assume (4') is a part of Lemma 1. As an aside, although we don't need any further properties of the unit, u , constructed above, it is also true that $u(1 - e') = (1 - e')$, $eu = e$, $e'u = e'$, and $(1 - e)u^{-1} = 1 - e'$.

The next two lemmas give us computational tools we will use to work inductively with suitable rings.

Lemma 2. *Let R be a suitable ring, and let $x_1 + x_2 + x_3 = 1$ be an equation in R . Suppose that x_1 is an idempotent. Then there are pair-wise orthogonal idempotents $e_1 \in Rx_1$, $e_2 \in Rx_2$, and $e_3 \in Rx_3$, such that $e_1 + e_2 + e_3 = 1$ and $x_1 \sim e_1$.*

Proof. Let $f = 1 - x_1$, and multiply by f on the left and right of $x_1 + x_2 + x_3 = 1$ to obtain $fx_2f + fx_3f = f$. Since corner rings in suitable rings are suitable [N, Proposition 1.10], fRf is suitable. Hence, there are orthogonal idempotents $f_2 \in fRf(fx_2f)$ and $f_3 \in fRf(fx_3f)$ summing to f (the identity in fRf). Write $f_2 = fr_2fx_2f$ and $f_3 = fr_3fx_3f$ for some $r_2, r_3 \in R$.

Let $e_2 = f_2r_2fx_2 \in Rx_2$ and let $e_3 = f_3r_3fx_3 \in Rx_3$. By an easy calculation we see that e_2 and e_3 are orthogonal idempotents. Let $e_1 = 1 - e_2 - e_3$, so e_1 is orthogonal to e_2 and e_3 , and we also obtain $e_1 + e_2 + e_3 = 1$.

We calculate

$$\begin{aligned} e_1x_1 &= (1 - e_2 - e_3)(1 - f) = 1 - e_2 - e_3 - f + e_2f + e_3f \\ &= e_1 - f + f_2 + f_3 = e_1 - f + f = e_1. \end{aligned}$$

So $e_1 \in Rx_1$. Finally, since $fe_2 = e_2$ and $fe_3 = e_3$, we see $x_1e_1 = x_1(1 - e_2 - e_3) = x_1$. \square

Lemma 3. *Let e and e' be idempotents in a ring R , with $e \sim e'$. Assume R has a linear, Hausdorff topology. Also assume that $e = \sum_{i \in I} g_i$ where $\{g_i\}_{i \in I}$ is a summable family of orthogonal idempotents. Then $\{e'g_i\}_{i \in I}$ is a summable family of orthogonal idempotents with $g_i \sim e'g_i$. Further, if $e' = ue$ then $e'g_i = ug_i$. Finally, if f is any idempotent orthogonal to e , then f is orthogonal to each g_i .*

Proof. Notice that $g_ie = g_i = eg_i$ and $ee' = e$. Therefore

$$(e'g_i)(e'g_j) = e'(g_ie)e'g_j = e'g_i(ee')g_j = e'g_i eg_j = e'g_i g_j = \delta_{i,j} e'g_i$$

so they are orthogonal idempotents. Also $g_i(e'g_i) = (g_ie)(e'g_i) = g_i eg_i = g_i$ and clearly $(e'g_i)g_i = e'g_i$. Thus $g_i \sim e'g_i$. If $e' = ue$ then $e'g_i = ueg_i = ug_i$. The final statement is another easy calculation. \square

It will turn out that we will be working with families of idempotents that are “almost” orthogonal, which we want to modify into truly orthogonal families. The following lemma gives us the mathematical framework to make this happen.

Lemma 4. *Let $\{e_i\}_{i \in I}$ be a summable family of idempotents in a ring R with a linear, Hausdorff topology, and assume I is well-ordered. Suppose that $e_i e_j \in J(R)$ whenever $i < j$, and that $\sum_{i \in I} e_i = u \in U(R)$. Then $\{u^{-1}e_i\}_{i \in I}$ is a family of orthogonal idempotents, summing to 1.*

Proof. Follows from [MM₁, Lemma 8]. □

Lemma 5. *Let $\{e_i\}_{i \in I}$ be a summable family of idempotents in a ring R with a linear, Hausdorff topology, and assume I is well-ordered. Put $e = \sum_{i \in I} e_i$ and suppose that $e_i e_j = 0$ whenever $i < j$. If $e^n r = 0$, for some $r \in R$ and some $n \in \mathbb{Z}_+$, then we have $e_i r = 0$ for all $i \in I$. In particular, $er = 0$.*

Proof. We proceed by induction. Since $e_i e_j = 0$ for $i < j$, this implies $e_1 e = e_1$ (where 1 is the first element of I). Therefore $e_1 e^n = e_1$, and so $e_1 r = e_1 e^n r = 0$. This finishes the base case.

Now, suppose that $e_i r = 0$ for all $i < \beta$. Then $er = \left(\sum_{i \geq \beta} e_i\right) r$. Again since $e_i e_j = 0$ for $i < j$, we have

$$e^{n-1} \left(\sum_{i \geq \beta} e_i\right) = e^{n-2} \left(\sum_{i < \beta} e_i + \sum_{i \geq \beta} e_i\right) \left(\sum_{i \geq \beta} e_i\right) = e^{n-2} \left(\sum_{i \geq \beta} e_i\right)^2 = \cdots = \left(\sum_{i \geq \beta} e_i\right)^n.$$

So,

$$0 = e_\beta e^n r = e_\beta e^{n-1} \left(\sum_{i \geq \beta} e_i\right) r = e_\beta \left(\sum_{i \geq \beta} e_i\right)^n r = e_\beta r.$$

This finishes the inductive step. It is now clear that $er = 0$ also. □

Lemma 6. *Let R be an exchange ring with a linear, Hausdorff topology. Then $J(R)$ is closed.*

Proof. This is [MM₁, Lemma 11]. The lemma they prove is for endomorphism rings, but the argument already works in this more general situation. □

Lemma 7. *Let R be a suitable ring, and put $\overline{R} = R/J(R)$. If $\varepsilon \in \overline{R}$ is an idempotent, then there is an idempotent $e \in R$ with $\overline{e} = \varepsilon$.*

Proof. Follows easily from [MM₁, Corollary 7]. □

§3. COUNTABLE EXCHANGE RINGS

The motivation for our first result comes from a simple construction showing that 2-exchange is equivalent to finite exchange for modules, based upon ideas in [N]. Unfortunately, the method fails when trying to pass to countable exchange. However, if one forces convergent limits of units to be units the proof can be made to work as follows.

Theorem 1. *Let R be a suitable ring with a nice topology. Also suppose that convergent limits of units are units. Then R is an \aleph_0 -exchange ring.*

Proof. Let $\{x_i\}_{i \in \mathbb{Z}_+}$ be a summable family of elements in R , with $\sum_{i=1}^{\infty} x_i = 1$. For notational ease, set $y_j = \sum_{i > j} x_i$. For each $j \in \mathbb{Z}_+$ we will construct elements $e_{i,j} \in Rx_i$ (for $i \leq j$), $f_j \in Ry_j$, and $v_j \in U(R)$ such that the following conditions hold: (1) $\{e_{1,j}, e_{2,j}, \dots, e_{j,j}, f_j\}$ is a family of orthogonal idempotents, summing to 1, and (2) $v_j e_{i,i} = e_{i,j}$ (for all $i \leq j$) and $v_j f_j = f_j$.

Set $v_1 = 1$. Since R is suitable, the equation $x_1 + y_1 = 1$ implies that there are orthogonal idempotents $e_{1,1} \in Rx_1$ and $f_1 \in Ry_1$ with $e_{1,1} + f_1 = 1$. It is easy to check that condition (1) holds for $j = 1$, and condition (2) holds trivially in this case. This finishes the base case. Suppose, by induction, we have fixed elements $e_{i,j} \in Rx_i$ (for all $i \leq j$), $f_j \in Ry_j$, and $v_j \in U(R)$ satisfying the conditions above, for each $j \leq n$. Writing $f_n = ry_n$ for some $r \in R$, we have

$$1 = e_{1,n} + \dots + e_{n,n} + f_n = (e_{1,n} + \dots + e_{n,n}) + rx_{n+1} + ry_{n+1}.$$

Lemma 2 allows us to pick pair-wise orthogonal idempotents

$$h_1 \in R(e_{1,n} + \dots + e_{n,n}), \quad h_2 \in Rrx_{n+1}, \quad h_3 \in Rry_{n+1}$$

with $h_1 + h_2 + h_3 = 1$ and $h_1 \sim \sum_{i=1}^n e_{1,n}$. By Lemma 1, property (4'), there exists $u_{n+1} \in U(R)$ such that $u_{n+1}(e_{1,n} + \dots + e_{n,n}) = h_1$ and $u_{n+1}f_n = f_n$. Putting $e_{i,n+1} = u_{n+1}e_{i,n} \in Rx_i$ (for $i \leq n$), $e_{n+1,n+1} = h_2 \in Rrx_{n+1}$, and $f_{n+1} = h_3 \in Rry_{n+1}$, Lemma 3 shows that condition (1) above holds.

By Lemma 1, property (5), $(e_{n+1,n+1} + f_{n+1})$ is right strongly isomorphic to f_n , hence $f_n e_{n+1,n+1} = e_{n+1,n+1}$ and $f_n f_{n+1} = f_{n+1}$. Putting $v_{n+1} = u_{n+1}v_n$, and remembering $u_{n+1}f_n = f_n$, we calculate

$$v_{n+1}f_{n+1} = (u_{n+1}v_n)(f_n f_{n+1}) = u_{n+1}v_n f_n f_{n+1} = u_{n+1}f_n f_{n+1} = f_n f_{n+1} = f_{n+1}$$

and similarly $v_{n+1}e_{n+1,n+1} = e_{n+1,n+1}$. Finally, for $i < n + 1$, $v_{n+1}e_{i,i} = u_{n+1}v_n e_{i,i} = u_{n+1}e_{i,n} = e_{i,n+1}$. Therefore, condition (2) holds. This finishes the inductive step.

So we have constructed elements $e_{i,j}$ (for $i \leq j$), f_j , and v_j satisfying the properties above, for all $j \in \mathbb{Z}_+$. Since $\{x_i\}_{i \in \mathbb{Z}_+}$ is summable, and the topology is left multiple summable, the family $\{e_{i,i}\}_{i \in \mathbb{Z}_+}$ is also summable. We put $\varphi = \sum_{i \in \mathbb{Z}_+} e_{i,i}$. We want to prove that φ is a unit in R .

Since $\lim_{n \rightarrow \infty} y_n = 0$, and the topology is linear, we have $\lim_{n \rightarrow \infty} f_n = 0$. Therefore,

$$\varphi = \sum_{i=1}^{\infty} e_{i,i} = \lim_{n \rightarrow \infty} \left(\sum_{i=1}^n e_{i,i} + f_n \right) = \lim_{n \rightarrow \infty} v_n^{-1} \left(\sum_{i=1}^n e_{i,n} + f_n \right) = \lim_{n \rightarrow \infty} v_n^{-1}.$$

Convergent limits of units are units, so φ is a unit.

Now, for $i < j$, we have $e_{i,i}e_{j,j} = v_j^{-1}v_j e_{i,i}e_{j,j} = v_j^{-1}e_{i,j}e_{j,j} = 0 \in J(E)$. So, by Lemma 4, $\{\varphi^{-1}e_{i,i}\}_{i \in \mathbb{Z}_+}$ is a summable, orthogonal set of idempotents, summing to 1. Finally, $\varphi^{-1}e_{i,i} \in Rx_i$, so R satisfies the definition of an \aleph_0 -exchange ring. \square

The converse of Theorem 1 is not true. For example, let $k = \mathbb{Q}$ and let $M_k = \mathbb{Q}_{\mathbb{Q}}^{(\mathbb{N})}$ be the countable vector space over \mathbb{Q} . Then E is isomorphic to the ring of $\mathbb{N} \times \mathbb{N}$ column-finite matrices over \mathbb{Q} . One can easily construct a limit of units in E which converges to a non-unit, and yet M has full exchange.

A natural question to ask is what convergent limits of units look like in general. We claim that in any ring with a linear, Hausdorff topology, a convergent limit of units is always a left non-zero-divisor. To see this, let $w = \lim_{i \in I} w_i$ with each w_i a unit, and with I well-ordered. Let $U \in \mathfrak{U}$ be an arbitrary, open (left ideal) neighborhood of 0. If $wr = 0$ then $\lim_{i \in I} w_i r = 0$ and so, in particular, for a large index N we have $w_N r \in U$. But U being a left ideal means $r = w_N^{-1} w_N r \in U$. Therefore $r \in \bigcap_{U \in \mathfrak{U}} U = (0)$. So $r = 0$.

Theorem 1 gives us the following chain of corollaries.

Corollary 1. *Let R be a suitable ring with a nice topology, and set $\overline{R} = R/J(R)$. If $\overline{R}_{\overline{R}}$ is cohopfian, then R is an \aleph_0 -exchange ring.*

Proof. Let $w = \lim_{i \in I} w_i$, where I is a well-ordered set, and $w_i \in U(R)$ for each $i \in I$. By Theorem 1, it suffices to show that w is a unit. An element $r \in R$ is a unit if and only if $\overline{r} \in \overline{R}$ is a unit. Further, by Lemma 6, we have $\overline{w} = \lim_{i \in I} \overline{w}_i$ in the quotient topology. Therefore it suffices to show that \overline{w} is a unit. Since \overline{w} is a limit of units it is a left non-zero-divisor. By [La₂, Exercise 4.16], $\overline{R}_{\overline{R}}$ is cohopfian if and only if all left non-zero-divisors are units. Thus \overline{w} is a unit. \square

Corollary 2. *Let R be a ring with a nice topology. If R is a Dedekind-finite, semi- π -regular ring then R is an \aleph_0 -exchange ring.*

Proof. All semi- π -regular rings are suitable rings. So, from the previous corollary, it suffices to show that $\overline{R}_{\overline{R}}$ is cohopfian.

Fix $x \in \overline{R}$ which is a left non-zero-divisor. Since \overline{R} is π -regular, fix some $n \geq 1$ such that x^n is (von Neumann) regular, say $x^n = x^n y x^n$ for some $y \in \overline{R}$. Then $x^n(1 - y x^n) = 0$. Since x is a left non-zero-divisor so is x^n . Therefore $1 = y x^n$, and so x is left-invertible. From the Dedekind-finiteness, which passes to \overline{R} , x is invertible. \square

Corollary 3. *Let R be a ring with a nice topology. If R is a strongly π -regular ring then R is an \aleph_0 -exchange ring.*

Proof. Strongly π -regular rings are always Dedekind-finite and π -regular. \square

§4. DEDEKIND-FINITE, REGULAR RINGS

When trying to push the proof of Theorem 1 up to full exchange one runs into problems when passing through limit ordinals. However, with the stronger hypothesis that R is a Dedekind-finite, regular ring, the proof goes through.

Theorem 2. *Let R be a ring with a nice topology. If R is a Dedekind-finite, regular ring then R is a full exchange ring.*

Proof. Let $\{x_i\}_{i \in I}$ be a summable collection of endomorphisms, summing to 1, with I an indexing set of arbitrary cardinality. Without loss of generality, we may assume that I is a well-ordered set, with first element 1, and last element κ . Put $y_j = \sum_{i > j} x_i$ and $y'_j = y_j + x_j = \sum_{i \geq j} x_i$.

For each $j \in I$ we will inductively construct elements $e_{i,j} \in Rx_i$ (for $i \leq j$), $f_j \in Ry_j$, and $v_j \in U(R)$ such that: (1) $\{e_{i,j} \ (\forall i \leq j), f_j\}$ is a family of orthogonal idempotents summing to 1, and (2) $v_j e_{i,i} = e_{i,j}$ (for each $i \leq j$) and $v_j f_j = f_j$.

Put $v_1 = 1$. Since R is regular it is suitable, and hence $x_1 + y_1 = 1$ implies that there are orthogonal idempotents $e_{1,1} \in Rx_1$ and $f_1 \in Ry_1$, which sum to 1. This completes the first step of our inductive definition. Now suppose (by trans-finite induction) that for all $j < \alpha$ we have constructed elements $e_{i,j}$ (for all $i \leq j$), f_j , and v_j satisfying the conditions above. We have two cases.

Case 1. α is not a limit ordinal.

In this case we proceed exactly as in the proof of Theorem 1. Writing $f_{\alpha-1} = ry_{\alpha-1}$ for some $r \in R$, we have

$$1 = \sum_{i < \alpha} e_{i,\alpha-1} + f_{\alpha-1} = \sum_{i < \alpha} e_{i,\alpha-1} + rx_{\alpha} + ry_{\alpha}.$$

Lemma 2 allows us to pick orthogonal idempotents

$$h_1 \in R \left(\sum_{i < \alpha} e_{i,\alpha-1} \right), \quad h_2 \in Rrx_{\alpha}, \quad h_3 \in Rry_{\alpha}$$

with $h_1 + h_2 + h_3 = 1$ and $h_1 \sim \sum_{i < \alpha} e_{i,\alpha-1}$. By Lemma 1, property (4'), there exists $u_{\alpha} \in U(R)$ such that $h_1 = u_{\alpha} \left(\sum_{i < \alpha} e_{i,\alpha-1} \right)$ and $u_{\alpha} f_{\alpha-1} = f_{\alpha-1}$. Putting $e_{i,\alpha} = u_{\alpha} e_{i,\alpha-1} \in Rx_i$ (for $i < \alpha$), $e_{\alpha,\alpha} = h_2 \in Rx_{\alpha}$, and $f_{\alpha} = h_3 \in Ry_{\alpha}$, then Lemma 3 implies that these are orthogonal idempotents. Also clearly

$$\sum_{i \leq \alpha} e_{i,\alpha} + f_{\alpha} = 1.$$

Therefore, condition (1) holds when $j = \alpha$. Checking that condition (2) holds for $v_{\alpha} = u_{\alpha} v_{\alpha-1}$ is done exactly as before. This completes the inductive definition of the elements we need, when α is not a limit ordinal.

Case 2. α is a limit ordinal.

This case is much harder and is where we really use the hypotheses on R . Setting $\varphi = \sum_{i < \alpha} e_{i,i}$, then since R is regular there is some $\psi \in E$ with $\varphi\psi\varphi = \varphi$, and in particular $p = 1 - \psi\varphi$ is an idempotent. Putting $\varphi' = \varphi + p$, we claim that φ' is a unit.

First, we do a few calculations. If $i < j < \alpha$, then $e_{i,i}e_{j,j} = v_j^{-1}v_j e_{i,i}e_{j,j} = v_j^{-1}e_{i,j}e_{j,j} = 0$. Also notice that $\varphi p = 0$. So, by Lemma 5, $e_{i,i}p = 0$ for all $i < \alpha$. Now, we show that φ' is a left non-zero-divisor. To see this, suppose first that $\varphi'\tau = 0$ for some $\tau \in R$. If $\varphi\tau = 0$ then $0 = \varphi'\tau = \varphi\tau + (1 - \psi\varphi)\tau = \tau$. So, we may assume $\varphi\tau \neq 0$, and in

particular there is a *smallest* index β with $e_{\beta,\beta}\tau \neq 0$. Then

$$0 = e_{\beta,\beta}(\varphi'\tau) = e_{\beta,\beta} \left(\sum_{i \in [\beta,\alpha)} e_{i,i}\tau + p\tau \right) = e_{\beta,\beta}\tau \neq 0$$

giving a contradiction. Thus, in all cases, φ' is a left non-zero-divisor. From our work in Corollary 2 we know that in a Dedekind-finite, regular ring any left non-zero-divisor is a unit. Therefore $\varphi' \in U(R)$.

For notational ease, put $v'_\alpha = (\varphi')^{-1}$. From our work above, we know that the decomposition $\varphi' = \sum_{i < \alpha} e_{i,i} + p$ satisfies the hypotheses of Lemma 4. This yields $\sum_{i < \alpha} v'_\alpha e_{i,i} + v'_\alpha p = 1$ where the summands are orthogonal idempotents. Put $e'_{i,\alpha} = v'_\alpha e_{i,i}$ and $f'_\alpha = v'_\alpha p$. An easy calculation shows that $f'_\alpha = p$, and in particular $v'_\alpha f'_\alpha = f'_\alpha$, which we will need later. We also claim $f_j f'_\alpha = f'_\alpha$ for all $j < \alpha$. To see this we compute

$$e_{i,j} f'_\alpha = v_j e_{i,i} f'_\alpha = v_j v_\alpha^{-1} v'_\alpha e_{i,i} f'_\alpha = v_j v_\alpha^{-1} e'_{i,\alpha} f'_\alpha = 0$$

and so

$$(1) \quad f_j f'_\alpha = \left(1 - \sum_{i \leq j} e_{i,j} \right) f'_\alpha = f'_\alpha.$$

Notice that we put hash marks on the idempotents we constructed. This is because they are not quite the ones we set out to construct. We need a few more modifications. The first problem with the idempotents we constructed above is that f'_α is not a left multiple of y'_α . We can fix this problem by finding a new idempotent in Ry'_α , which we will eventually call f''_α , which is right strongly isomorphic to f'_α . The construction is as follows:

Since R is regular, the principal right ideal $y'_\alpha f'_\alpha R$ is generated by an idempotent g_α , due to [La₂, Theorem 4.23]. So there is some $z_\alpha \in R$ with $g_\alpha = y'_\alpha f'_\alpha z_\alpha$, where we may assume $z_\alpha g_\alpha = z_\alpha$. Also note,

$$(2) \quad g_\alpha y'_\alpha f'_\alpha = y'_\alpha f'_\alpha.$$

By definition, for $i < \alpha$ we have $f_i \in Ry_i$, and so we can fix elements $r_i \in R$ with $f_i = r_i y_i$. For use shortly, we also note

$$(3) \quad \lim_{i \rightarrow \alpha} y_i = y'_\alpha.$$

Set $r'_\alpha = f'_\alpha z_\alpha$. Then using equations 1 and 3 above, along with left linearity, we have the following alternate definition of r'_α :

$$(4) \quad r'_\alpha = f'_\alpha z_\alpha = \lim_{i \rightarrow \alpha} f'_\alpha z_\alpha = \lim_{i \rightarrow \alpha} f_i f'_\alpha z_\alpha = \lim_{i \rightarrow \alpha} r_i y_i f'_\alpha z_\alpha = \lim_{i \rightarrow \alpha} r_i y'_\alpha f'_\alpha z_\alpha = \lim_{i \rightarrow \alpha} r_i g_\alpha.$$

We define $f''_\alpha = r'_\alpha y'_\alpha = f'_\alpha z_\alpha y'_\alpha$. We first do the easy computation to show that this is an idempotent:

$$f''_\alpha f''_\alpha = f'_\alpha z_\alpha y'_\alpha f'_\alpha z_\alpha y'_\alpha = f'_\alpha (z_\alpha y'_\alpha f'_\alpha z_\alpha) y'_\alpha = f'_\alpha (z_\alpha g_\alpha) y'_\alpha = f'_\alpha z_\alpha y'_\alpha = f''_\alpha.$$

Using equations 1 through 4 above, we compute

$$\begin{aligned} f''_\alpha f'_\alpha &= r'_\alpha y'_\alpha f'_\alpha = \left(\lim_{i \rightarrow \alpha} r_i g_\alpha \right) y'_\alpha f'_\alpha = \lim_{i \rightarrow \alpha} r_i (g_\alpha y'_\alpha f'_\alpha) \\ &= \lim_{i \rightarrow \alpha} r_i y'_\alpha f'_\alpha = \lim_{i \rightarrow \alpha} r_i y_i f'_\alpha = \lim_{i \rightarrow \alpha} f_i f'_\alpha = \lim_{i \rightarrow \alpha} f'_\alpha = f'_\alpha \end{aligned}$$

Also, clearly, $f'_\alpha f''_\alpha = f''_\alpha$.

We have shown $f'_\alpha \sim f''_\alpha$. Therefore the equivalence of properties (1) and (5) in Lemma 1 implies $(1 - f'_\alpha) \sim (1 - f''_\alpha)$. So, again by Lemma 1, property (4'), pick some unit v''_α such that $v''_\alpha(1 - f'_\alpha) = 1 - f''_\alpha$ and $v''_\alpha f'_\alpha = f''_\alpha$. Set $e''_{i,\alpha} = v''_\alpha e'_{i,\alpha}$, for $i < \alpha$. We have $\sum_{i < \alpha} e''_{i,\alpha} + f''_\alpha = 1$, and $\{e''_{i,\alpha} \ (\forall i < \alpha), f''_\alpha\}$ is a summable family of orthogonal idempotents by Lemma 3.

With all the machinery we have built up, it is now an easy matter to construct $e_{i,\alpha}$ (for all $i \leq \alpha$), f_α , and v_α . To do so, notice we have the equation

$$1 = \sum_{i < \alpha} e''_{i,\alpha} + f''_\alpha = \sum_{i < \alpha} e''_{i,\alpha} + r'_\alpha x_\alpha + r'_\alpha y_\alpha.$$

Now use exactly the same ideas as in Case 1 to construct the elements we need. However, there is one non-trivial step. We cannot put $v_\alpha = u_\alpha v_{\alpha-1}$ since α has no predecessor. Instead, we must put $v_\alpha = u_\alpha v''_\alpha v'_\alpha$. It is clear that $v_\alpha e_{i,i} = e_{i,\alpha}$ for $i < \alpha$, so we just need to see that left multiplication by v_α acts as the identity on $e_{\alpha,\alpha}$ and f_α . First, remember $f'_\alpha = v'_\alpha f''_\alpha$. Second, we chose v''_α so that $v''_\alpha f'_\alpha = f''_\alpha$. Third, just as in Case 1 where u_α was chosen so that $u_\alpha f_{\alpha-1} = f_{\alpha-1}$, here we can choose u_α so that $u_\alpha f''_\alpha = f''_\alpha$. Finally, $e_{\alpha,\alpha}$ and f_α are both fixed by left multiplication by f''_α and f'_α since $(e_{\alpha,\alpha} + f_\alpha) \sim f''_\alpha \sim f'_\alpha$. Therefore,

$$v_\alpha f_\alpha = (u_\alpha v''_\alpha v'_\alpha)(f'_\alpha f_\alpha) = u_\alpha (v''_\alpha v'_\alpha f'_\alpha) f_\alpha = u_\alpha f'_\alpha f_\alpha = u_\alpha f_\alpha = u_\alpha (f''_\alpha f_\alpha) = f''_\alpha f_\alpha = f_\alpha$$

and similarly, $v_\alpha e_{\alpha,\alpha} = e_{\alpha,\alpha}$. This finishes Case 2.

By trans-finite induction, we have constructed the elements we wanted for all $j \in I$. To finish the theorem, let $e_i = e_{i,\kappa}$ for all $i \leq \kappa$. Then $\{e_i\}_{i \in I}$ is a summable family of orthogonal idempotents, summing to 1 (since $f_\kappa \in Ry_\kappa = (0)$), with $e_i \in Rx_i$ for each $i \in I$. This completes the proof. \square

Corollary 4. *Let R be a ring with a nice topology. If R is unit-regular then R is a full exchange ring.*

Proof. Unit-regular rings are always regular and Dedekind-finite. \square

We did not state Theorem 2 in full generality so as not to become bogged down with the details, and in an effort to make the proof feel more natural. Now that the basic construction is finished we can work in a more general setting.

Theorem 3. *Let R be a ring with a nice topology. If R is π -regular and R_R has (C_2) then R is a full exchange ring.*

Proof. We need only look at how the hypothesis of regularity was used in Theorem 2. First was the fact that regularity implied suitability. But R is suitable since R is π -regular.

Second, we needed φ to be regular. We know it is π -regular, and so there is some $n \geq 1$, and some $\psi \in R$, with $\varphi^n = \varphi^n \psi \varphi^n$. Thus $\varphi^n(1 - \psi \varphi^n) = 0$. By Lemma 5, $\varphi(1 - \psi \varphi^n) = 0$. In other words, $\varphi = \varphi(\psi \varphi^{n-1})\varphi$. Therefore, φ is still a regular element.

Third, we needed the fact that regularity plus Dedekind-finiteness forces left non-zero-divisors to be units, but this also holds in the case R is π -regular.

Finally, R being regular told us that $y'_\alpha f'_\alpha R$ was generated by an idempotent. We claim that $y'_\alpha f'_\alpha R \cong f'_\alpha R$, and therefore $y'_\alpha f'_\alpha R$ will be generated by an idempotent because of the (C_2) hypothesis. It suffices to show that if $y'_\alpha f'_\alpha r = 0$ then $f'_\alpha r = 0$. Using equations 1 and 3 above, we see $f'_\alpha r = \lim_{i \rightarrow \alpha} f_i f'_\alpha r = \lim_{i \rightarrow \alpha} r_i y_i f'_\alpha r = \lim_{i \rightarrow \alpha} r_i y_\alpha f'_\alpha r = 0$. \square

§5. LIFTING THROUGH THE JACOBSON RADICAL

Mohamed and Müller have shown in [MM₂] that if M is a module such that $E/J(E)$ is regular and abelian, with idempotents lifting modulo $J(E)$, then M has full exchange. In particular, they use this to establish that continuous modules have exchange. Similarly, one way of further generalizing the results of the previous sections is to try and lift the argument through the Jacobson radical. The argument is actually quite easy.

Theorem 4. *Let R be a ring with a nice topology. Assume that R_R has (C_2) and R is a Dedekind finite, semi- π -regular ring. Then R is a full exchange ring.*

Proof. First, notice that R is suitable. If one works through the proofs of Theorems 2 and 3, the only other point which needs some modification is the choice of the idempotent p . There are two properties we need p to satisfy. First, we need $\varphi p = 0$, so that the calculation showing $p = f'_\alpha$ will work, and also so $e_{i,i} p = 0$ for all $i < \alpha$. Second, we need $\varphi' = \varphi + p$ to be a unit.

By Lemma 6, we have that $\{\bar{e}_{i,i}\}_{i < \alpha}$ is summable in the quotient topology of $R/J(R)$, summing to $\bar{\varphi}$. Since $R/J(R)$ is π -regular, the argument in Theorem 3 shows that $\bar{\varphi}$ is regular. Hence, there is some $\psi \in R$ with $\varphi - \varphi \psi \varphi \in J(R)$. Since idempotents lift modulo $J(R)$, and since $1 - \psi \varphi$ is an idempotent modulo $J(R)$, we can pick an idempotent $\tilde{p} \in R$ (not quite the one we want) with $\tilde{p} - (1 - \psi \varphi) \in J(R)$. Put $\tilde{\varphi} = \varphi + \tilde{p}$.

We want to show $\tilde{\varphi}$ is a unit in R , and so it suffices to show that $\bar{\tilde{\varphi}}$ is a unit in $R/J(R)$. But because of how \tilde{p} was chosen, the same argument in Theorems 2 and 3, which showed φ' was a unit, will now show that $\bar{\tilde{\varphi}}$ is a unit. To make things explicit, we will repeat the argument here.

Since $R/J(R)$ is π -regular and Dedekind-finite, it suffices to show that $\bar{\tilde{\varphi}}$ is a left non-zero-divisor. Suppose $\bar{\tilde{\varphi}}\bar{\tau} = 0$ for some $\tau \in R$. If $\bar{\varphi}\bar{\tau} = 0$, then since $\tilde{p} - (1 - \psi \varphi) \in J(R)$, we have $0 = \bar{\tilde{\varphi}}\bar{\tau} = \overline{\varphi\tau + (1 - \psi \varphi)\tau} = \bar{\tau}$. Therefore, we may assume $\bar{\varphi}\bar{\tau} \neq 0$, and in

particular there is a smallest index β , with $\overline{e_{\beta,\beta}\tau} \neq 0$. Now, $\overline{\varphi\tilde{p}} = 0$, and so Lemma 5 implies that $\overline{e_{i,i}\tilde{p}} = 0$ for all $i < \alpha$. Therefore, working modulo $J(R)$, we calculate

$$0 \equiv e_{\beta,\beta}(\varphi'\tau) \equiv e_{\beta,\beta} \left(\sum_{i \in [\beta,\alpha)} e_{i,i}\tau + p\tau \right) \equiv e_{\beta,\beta}\tau \notin 0 + J(R).$$

This contradiction shows that $\tilde{\varphi}$ is a left non-zero-divisor, and hence a unit.

In our work above we found that $e_{i,i}\tilde{p} \in J(R)$ for $i < \alpha$. Then, by Lemma 4, the collection $\{(\tilde{\varphi})^{-1}e_{i,i} \ (\forall i < \alpha), (\tilde{\varphi})^{-1}\tilde{p}\}$ consists of orthogonal idempotents, summing to 1. Put $p = (\tilde{\varphi})^{-1}\tilde{p}$. Since $(\tilde{\varphi})^{-1}e_{i,i}p = 0$ we have $e_{i,i}p = 0$, and in particular $\varphi p = 0$.

Set $\varphi' = \varphi + p$. Suppose that $\varphi\tau = 0$ for some $\tau \in R$. Then,

$$\tau = \left(\sum_{i < \alpha} (\tilde{\varphi})^{-1}e_{i,i} + p \right) \tau = (\tilde{\varphi})^{-1}\varphi\tau + p\tau = p\tau = \varphi'\tau.$$

Notice that we can push this equation down to $R/J(R)$. Showing $\varphi' = \varphi + p$ is a unit is now a simple matter by copying the ideas used in the proof that $\tilde{\varphi}$ is a unit. \square

One also has another way to lift the argument through the radical.

Corollary 5. *Let R be a ring with a nice topology. If R is a Dedekind-finite, semi-regular ring then R is a full exchange ring.*

Proof. Let $\{x_i\}_{i \in I}$ be a summable family of idempotents, summing to 1. Let I be well-ordered as usual. Putting $\overline{R} = R/J(R)$, then we see by Lemma 6 that \overline{R} is a topological ring in the quotient topology with a linear, Hausdorff topology. Further, $\{\overline{x_i}\}_{i \in I}$ is a summable family summing to 1, and is left-multiple summable, since $\{x_i\}_{i \in I}$ is. Therefore, the same argument as used in the proof of Theorem 2 shows that we can find orthogonal idempotents $\varepsilon_i \in \overline{R}\overline{x_i}$ summing to 1.

By Lemma 7, we can lift each ε_i to an idempotent $e_i \in Rx_i$. These are still summable idempotents, summing to a unit (since, modulo $J(R)$, they sum to 1). Letting $u = \sum_{i \in I} e_i$, then Lemma 4 says that $\{u^{-1}e_i\}_{i \in I}$ is a summable family of orthogonal idempotents summing to 1. Clearly, $u^{-1}e_i \in Rx_i$, so we are done. \square

Using the same ideas, we also have

Corollary 6. *Let R be a ring with a nice topology. If R is a Dedekind-finite, semi- π -regular ring, and $\overline{R}_{\overline{R}}$ has (C_2) , then R is a full exchange ring.*

§6. EXCHANGE MODULES

What do the previous theorems say concerning finite exchange modules? We have the following unsettling fact, motivated by [La₁, Proposition 8.11].

Lemma 8. *Let M_k be a module, and $E = \text{End}(M_k)$, as usual. If E_E is cohopfian, or respectively has (C_2) , then so does M . The converses do not hold.*

Proof. First, suppose that E_E is cohopfian. Let $x \in E$ be an injective endomorphism on M . If $xr = 0$ for some $r \in E$, then $xr(m) = 0$ for all $m \in M$. But, x being injective implies $r(m) = 0$ for all $m \in M$. Therefore, $r = 0$. Since r was arbitrary, x is a left non-zero-divisor. Therefore, since E_E is cohopfian, x is a unit. This shows that M is cohopfian.

Now instead suppose that E_E has (C_2) . Consider the situation where $N' \cong N \subseteq^\oplus M$. Let $e \in E$ be an idempotent with $e(M) = N$, and let $\varphi : N \rightarrow N'$ be an isomorphism. Without loss of generality, we may assume $\varphi \in E$ by setting φ equal to 0 on $(1 - e)(M)$.

Consider the map, $eE \rightarrow \varphi eE$, given by left multiplication by φ . Clearly this is surjective. To show injectivity, suppose that $\varphi er = 0$ for some $r \in E$. Then $\varphi er(m) = 0$ for all $m \in M$. In particular, $\varphi(er(M)) = 0$. But $er(M) \subseteq e(M)$ and φ is injective on $e(M) = N$, therefore $er(M) = 0$. But then $er = 0$. This shows injectivity.

Thus φeE is isomorphic to eE , a direct summand of E_E . Therefore φeE is generated by an idempotent, say f . Clearly $f\varphi e = \varphi e$, and $f = \varphi ey$ for some $y \in E$. So $f(M) = \varphi ey(M) \subseteq \varphi e(M) = N'$, and $f(M) \supseteq f(\varphi e(M)) = \varphi e(M) = N'$. Therefore $N' = f(M)$ is a direct summand.

A single counter-example will show that both converses do not hold. Let $k = \mathbb{Z}$ and let M be the Prüfer p -group, for any prime p . Then E is isomorphic to the ring of p -adic integers. M is cohopfian while E is not, by [La₁, Proposition 8.11]. Notice that the only idempotents in E are 0 and 1. Thus, the only direct summands in either M_k or E_E are the trivial ones. One easily sees that multiplication by p yields $pE \cong E_E$, but pE is not a summand. Therefore E_E does not have the (C_2) property. On the other hand, any submodule isomorphic to M must contain elements killed by multiplication by p , and hence must equal M . Thus, all submodules of M isomorphic to M are summands, and all submodules of M isomorphic to (0) equal (0) . Hence M has the (C_2) property. \square

Due to this lemma, it would appear that one could not work with the weaker notion of a cohopfian module and hope to prove a theorem analogous to Theorem 1. However, in endomorphism rings, limits of units are very special.

Theorem 5. *Let M be a cohopfian module with finite exchange. Then M has countable exchange.*

Proof. In the endomorphism ring, E , a limit of units must be an injective endomorphism (since nothing in the limit process has a kernel). But then the cohopfian condition forces this endomorphism to be an isomorphism, or in other words a unit in E . Thus convergent limits of units are units. So M has countable exchange from Theorem 1. \square

Can one also tweak Theorem 4 so we are working with the weaker hypothesis that M has the (C_2) property? The answer is yes.

Theorem 6. *Let M be a module with the (C_2) property, and a Dedekind-finite, semi- π -regular endomorphism ring. Then M has full exchange.*

Proof. Following Theorem 4, with $R = E$, the only thing we need to do differently is find an idempotent $f''_\alpha \in Ey'_\alpha$ with $f'_\alpha \sim f''_\alpha$.

Consider the map $y'_\alpha : f'_\alpha(M) \rightarrow y'_\alpha f'_\alpha(M)$, given by left-multiplication by y'_α . It is clearly surjective. We have $f'_\alpha = \lim_{i \rightarrow \alpha} f_i f'_\alpha = \lim_{i \rightarrow \alpha} r_i y_i f'_\alpha = \lim_{i \rightarrow \alpha} r_i y'_\alpha f'_\alpha$, and so the map above must also be injective. From the (C_2) hypothesis, we have that $y'_\alpha f'_\alpha(M) = g_\alpha(M)$ for some idempotent g_α .

Define r'_α by the rule $r'_\alpha|_{(1-g_\alpha)(M)} = 0$ and $r'_\alpha|_{g_\alpha(M)=y'_\alpha f'_\alpha(M)} = \lim_{i \rightarrow \alpha} r_i$. While it is true that $\lim_{i \rightarrow \alpha} r_i$ does not necessarily converge in general, it does converge on $y'_\alpha f'_\alpha(M)$ since $\lim_{i \rightarrow \alpha} r_i y'_\alpha f'_\alpha(m) = \lim_{i \rightarrow \alpha} r_i y_i f'_\alpha(m) = \lim_{i \rightarrow \alpha} f_i f'_\alpha(m) = f'_\alpha(m)$.⁴

We put $f''_\alpha = r'_\alpha y'_\alpha$. We first check that it is an idempotent. Given $m \in M$, we can write $y'_\alpha(m) = g_\alpha y'_\alpha(m) + (1 - g_\alpha)y'_\alpha(m) = y'_\alpha f'_\alpha(m') + (1 - g_\alpha)y'_\alpha(m)$ for some $m' \in M$. Then,

$$\begin{aligned} f''_\alpha f''_\alpha(m) &= r'_\alpha y'_\alpha r'_\alpha (y'_\alpha f'_\alpha(m') + (1 - g_\alpha)y'_\alpha(m)) = r'_\alpha y'_\alpha (r'_\alpha y'_\alpha f'_\alpha(m')) \\ &= r'_\alpha y'_\alpha \left(\lim_{i \rightarrow \alpha} r_i y_i f'_\alpha \right) (m') = r'_\alpha y'_\alpha \left(\lim_{i \rightarrow \alpha} f_i f'_\alpha \right) (m') = r'_\alpha y'_\alpha f'_\alpha(m') \\ &= r'_\alpha (y'_\alpha f'_\alpha(m') + (1 - g_\alpha)y'_\alpha(m)) = r'_\alpha y'_\alpha(m) = f''_\alpha(m). \end{aligned}$$

So $f''_\alpha f''_\alpha = f''_\alpha$. A similar computation shows that f''_α and f'_α are right strongly isomorphic. The rest of the proof follows Theorem 4. \square

Theorem 2, and Corollaries 1 through 6 immediately translate over to the endomorphism ring case. In particular, we have:

Corollary 7. *If M has a Dedekind-finite, semi- π -regular endomorphism ring, then M has countable exchange. If, further, the endomorphism ring is semi-regular, then M has full exchange.*

§7. FINAL REMARKS

In [Ni], we define what we call *finitely complemented* modules. These are modules whose direct summands have only finitely many complement summands. We showed that a finitely complemented module with a regular endomorphism ring has full exchange. We claim that using the methods derived above, one can remove the condition that E is regular, and replace it with M having finite exchange and (C_2) .

There is another class of modules we can apply these techniques to; namely, square-free modules. Suppose that M is a square-free module with finite exchange. Mohamed and Müller have shown that $E/J(E)$ is abelian, [MM₁, Lemmata 11 and 15]. In particular, the element $\varphi = \sum_{i < \alpha} e_{i,i}$, used in our proof above, is an idempotent in $E/J(E)$. [Since $\bar{e}_{i,i} \bar{e}_{j,j} = 0$ for $i < j$, and since idempotents commute in an abelian ring, $\bar{\varphi}$ is a sum

⁴One should now also check that r'_α is a well-defined homomorphism, which we leave to the reader.

of orthogonal idempotents, and hence is an idempotent.] Since idempotents lift modulo $J(E)$ (because E is suitable) we can lift $\overline{1 - \varphi}$ to an idempotent \tilde{p} , as before. Notice that $\tilde{\varphi} = \varphi + \tilde{p}$ is a unit since $\tilde{\varphi}$ is congruent to 1 modulo $J(E)$. One chooses p as in Theorem 4. Finally, if M has (C_2) we can proceed as in Theorem 6 to show full exchange for M . However, for square-free modules, the (C_2) property is equivalent to cohopfianness. So what we have shown is that a cohopfian, square-free module with finite exchange has full exchange.

As far as we know, the only classes of modules where it is known that finite exchange implies countable exchange, but not known if this further implies full exchange, are square-free modules, cohopfian modules, and finitely complemented modules.

REFERENCES

- [CJ] P. Crawley and B. Jónsson: *Refinements for infinite direct decompositions of algebraic systems*, Pacific J. Math. **14**(1964), 797-855.
- [La₁] T. Y. Lam: *A Crash Course on Stable Range, Cancellation, Substitution, and Exchange*, Preprint, 2003.
- [La₂] T. Y. Lam: *A First Course in Noncommutative Rings*, 2nd edition, Graduate Texts in Math., Vol. **131**, Springer-Verlag, Berlin-Heidelberg-New York, 2001.
- [MM₁] S. H. Mohamed and B. J. Müller: *\aleph -exchange rings*, in “Abelian Groups, Module Theory, and Topology”, 311-317, Lecture Notes in Pure and Applied Math., Vol. **201**, Marcel Dekker, New York, 1998.
- [MM₂] S. H. Mohamed and B. J. Müller: *Continuous modules have the exchange property*, in “Abelian Group Theory” (Perth, 1987), 285-289, Contemp. Math., Vol. **87**, Amer. Math. Soc., Providence, RI, 1989.
- [MM₃] S. H. Mohamed and B. J. Müller: *On the exchange property for quasi-continuous modules*, in “Abelian Groups and Modules” (Padova, 1994) 367-372, Math. Appl. **343**, Kluwer Acad. Publ., Dordrecht, 1995.
- [N] W. K. Nicholson: *Lifting idempotents and exchange rings*, Trans. A.M.S. **229**(1977), 269-278.
- [Ni] P. Nielsen: *Abelian exchange modules*, Preprint, 2003.
- [OR] K. Oshiro and S. T. Rizvi: *The exchange property of quasi-continuous modules with the finite exchange property*, Osaka J. Math. **33**(1996), 217-234.
- [Wa] R. B. Warfield: *Exchange rings and decompositions of modules*, Math. Ann. **199**(1972), 31-36.
- [ZZ] B. Zimmermann-Huisgen and W. Zimmermann: *Classes of Modules with the Exchange Property*, J. Algebra **88**(1984), 416-434.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CALIFORNIA, BERKELEY, CA 94720

E-mail address: pace@math.berkeley.edu